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Analysis on the Steady-State Coherent Synchrotron Radiation with Strong Shielding

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Abstract

There are several papers concerning shielding of coherent synchrotron radiation (CSR) emitted by a Gaussian line charge on a circular orbit centered between two parallel conducting plates. Previous asymptotic analyses in the frequency domain show that shielded steady-state CSR mainly arises from harmonics in the bunch frequency exceeding the threshold harmonic for satisfying the boundary conditions at the plates. In this paper we extend the frequency-domain analysis into the regime of strong shielding, in which the threshold harmonic exceeds the characteristic frequency of the bunch. The result is then compared to the shielded steady-state CSR power obtained using image charges.

1 INTRODUCTION

There have been several studies [1, 2, 3] concerning shielding of coherent synchrotron radiation (CSR) emitted by a Gaussian line charge on a circular orbit centered between two parallel conducting plates. Nodvick and Saxon [1] developed an exact expression for the power radiated by a bunch in steady state, written as a summation over all harmonics of the radiated power. Using the asymptotic behavior of the Bessel functions in these radiated-power harmonics, Kheifets and Zotter [2] recently developed a simple expression for the shielded CSR power as a function of beam and machine parameters. In an alternative derivation, Murphy, Krinsky, and Gluckstern [3] obtain the CSR power by including image charges from the parallel plates to calculate the CSR-induced steady-state longitudinal electric force across the bunch.

According to Ref. [2], shielded CSR is important for harmonic numbers n in the range $n_{th} < n < n_c$, where

$$n_{th} = \sqrt{2/3}(\pi\rho/h)^{3/2}, \qquad n_c = \rho/\sigma_s. \tag{1}$$

Here, ρ and h denote the radius of the circular orbit and the plate separation, respectively, σ_s denotes the root-mean-square bunch length, n_{th} is the threshold harmonic for satisfying the boundary conditions at the plates, and n_c is the characteristic harmonic number below which the radiation will be coherent. However, according to Fig. 9 of Ref. [3], which compares the shielded CSR power calculated using the image charge method with that given in Ref. [2], there is an evident discrepancy. For $n_{th} > n_c$, i.e., for strong shielding, the result given by Ref. [2] underestimates the shielded CSR power considerably.

In this paper, we modify the analysis of Ref. [2] using power harmonics in Ref. [1] to obtain a result for the steady-state CSR power in agreement with the exact result for $n_{th} > n_c$. We show that for this parameter range the shielded CSR power is not always negligibly small.

2 PREVIOUS RESULTS

We consider a line-charge bunch moving on a circular orbit of radius ρ with angular frequency ω_0 in the center plane between two perfectly conducting plates at $z = \pm h/2$. According to Kheifets and Zotter [2], based on the work of Nodvick and Saxon, the power radiated by a single relativistic electron in the nth harmonic is

$$P_n = \frac{4\pi n e^2\omega_0}{h}\sum_{p=1,3}^{p<nh/\pi\rho}\left[J_n'^2(\gamma_p\rho) + \frac{g_p^2}{n^2-g_p^2}J_n^2(\gamma_p\rho)\right] \tag{2}$$

where $\gamma_p\rho = \sqrt{n^2-g_p^2}$ with $g_p = p\pi\rho/h$, and p is the index of eigenfunctions satisfying the boundary condition at the plates. Only propagating modes, i.e., those satisfying $n > g_p$, contribute significantly to the power P_n; therefore, when $\pi\rho/h \gg 1$ it suffices to use asymptotic expressions of the Bessel functions for large n:

$$\begin{aligned} J_n(\gamma_p\rho) &\simeq \frac{1}{\sqrt{3}\pi}\left(\frac{g_p}{n}\right)K_{1/3}(g_p^3/3n^2), \\ J_n'(\gamma_p\rho) &\simeq \frac{1}{\sqrt{3}\pi}\left(\frac{g_p}{n}\right)^2K_{2/3}(g_p^3/3n^2). \end{aligned} \tag{3}$$

The modified Bessel functions are appreciable only when $n > g_p^{3/2} \gg g_p$. Consequently, for $\omega_0 = c/\rho$, the power harmonic of N electrons reduces to

$$P_n = \frac{4N^2e^2c}{3\pi\rho h}\sum_{p=1,3}^{p<nh/\pi\rho}A(n,p) \tag{4}$$

where

$$A(n,p) \equiv \frac{g_p^4}{n^3}\left[K_{1/3}^2\left(\frac{g_p^3}{3n^2}\right) + K_{2/3}^2\left(\frac{g_p^3}{3n^2}\right)\right]. \tag{5}$$

The CSR power generated by a relativistic Gaussian line charge with N electrons is

$$P_{coh} = \sum_{n=0}^{\infty}P_n e^{-(n\sigma_s/\rho)^2}. \tag{6}$$

In Ref. [2], it is assumed that the CSR power comprises mainly harmonics P_n for which $n > n_{th}$. Changing the summation over p into integration gives

$$P_n = \frac{C_0N^2e^2c}{\rho}n^{1/3} \qquad (n_{th} < n < n_c) \tag{7}$$

where $C_0 = 2C/3^{1/3}\pi^2$, with $C \simeq 3.68$. The total CSR power is then obtained by replacing the summation over n by an integration over $x = (n\sigma_s/\rho)^2$,

$$P_{coh} \approx \frac{C_0 N^2 e^2 c}{2\rho}\left(\frac{\rho}{\sigma_s}\right)^{4/3} F(x_{th}),$$
$$F(x_{th}) = \int_{x_{th}}^{x_c} dx\, x^{-1/3} e^{-x} \approx \Gamma(2/3, x_{th}) \qquad (8)$$

with $x_{th} = (n_{th}/n_c)^2$, $x_c = 4\pi$, and $\Gamma(\nu, x)$ denoting an incomplete gamma function.

The free space CSR power is obtained by setting x_{th} to 0,

$$P^{(0)}_{coh} = \frac{N^2 e^2 c}{(\rho^2\sigma_s^4)^{1/3}} \frac{F(0)C_0}{2} \qquad (9)$$

where $F(0)C_0/2 \approx 0.35$. This result agrees with that obtained by Schiff [4],

$$P^{(0)}_{coh} = \frac{N^2 e^2 c}{(\rho^2\sigma_s^4)^{1/3}} \frac{3^{1/6}[\Gamma(2/3)]^2}{2\pi} \qquad (10)$$

in that $3^{1/6}[\Gamma(2/3)]^2/2\pi \approx 0.35$. The formalism of Eq. (8) therefore sets the ratio of shielded CSR to that of free space as

$$P_{coh}/P^{(0)}_{coh} = F(x_{th})/\Gamma(2/3). \qquad (11)$$

3 MODIFIED ANALYSIS

We will now show that the results of the previous section are applicable only for weak shielding, namely, $n_{th} \ll n_c$, due to the replacement of the sum over p by an integral and the discarding of the contribution of harmonics $n < n_{th}$. We do so by first showing that, for each pth mode satisfying the boundary conditions at the plates, there is a threshold harmonic $n^{(p)}_{th} = p^{3/2} n_{th}$, where n_{th} given in Eq. (1) is the threshold harmonic for $p = 1$, i.e., $n_{th} \equiv n^{(p=1)}_{th}$. The pth mode contributes significantly to the CSR power only when $n^{(p)}_{th} < n_c$, or $p < (n_c/n_{th})^{2/3}$. Using integration over p on the interval $[0, \infty)$ to derive Eq. (7) from Eq. (4) is valid only if $n_{th} \ll n_c$, which is the quasi-free-space, weak-shielding situation. However, if the parameters do not satisfy $n_{th} \ll n_c$, only the first few p modes contribute to P_{coh} in Eq. (6). In particular, when $n_{th} > n_c$, only the $p = 1$ mode contributes significantly to P_{coh}, and in this case Eq. (8) is manifestly inaccurate.

We begin our analysis by showing that it is permissible to exchange the order of the summations over p and n, which then allows an explicit calculation of the contribution of the pth mode to the CSR power. The asymptotic behaviors of the modified Bessel functions are

$$K_\nu(z) \sim \begin{cases} \sqrt{\dfrac{\pi}{2z}}\, e^{-z} & \text{(large } z\text{)} \\ \dfrac{1}{2}\Gamma[\nu](z/2)^{-\nu} & (z \to 0, \operatorname{Re}\nu \neq 0) \end{cases} . \qquad (12)$$

Numerically one finds that the asymptotic behavior of $K_\nu(z)$ for large z is good to 10% at $z = 1/2$ for $\nu = 1/3, 2/3$. Therefore, the modified Bessel functions in Eq. (5) are exponentially small when $g_p^3/3n^2 \gg 1/2$ or $n \ll n^{(p)}_{th}$, and their contributions to P_n are then negligible. This typifies the nonpropagating modes, i.e., those for which $p \geq nh/\pi\rho$, because $g_p^3/3n^2 \geq n/3 \gg 1$ for large n. Consequently one can extend the summation over p to infinity in Eq. (4) without sacrificing accuracy, and therefore exchange the orders of summation for indices p and n in combining Eqs. (4) and (6) to obtain

$$P_{coh} = \frac{4N^2 e^2 c}{3\pi\rho h} \sum_{p=1,3}^{\infty} I(p), \qquad (13)$$

with $I(p)$ denoting the contribution of the pth mode to the coherent radiated power:

$$I(p) = \sum_{n=0}^{\infty} A(n,p) e^{-(n/n_c)^2}. \qquad (14)$$

To obtain a closed-form expression for P_{coh}, we first calculate the asymptotic form of $I(p)$. After defining

$$x^{(p)}_{th} = (n^{(p)}_{th}/n_c)^2, \quad x = (n/n_c)^2, \qquad (15)$$

we note that, provided $n_c \gg 1$ and $n^{(p)}_{th} \gg 1$, the summation over n in Eq. (14) can be replaced by an integral:

$$I(p) \simeq \frac{3g_p}{4} \int_0^\infty f^{(p)}(x)dx, \qquad (16)$$

with

$$f^{(p)}(x) = \left[K^2_{1/3}\left(\frac{x^{(p)}_{th}}{2x}\right) + K^2_{2/3}\left(\frac{x^{(p)}_{th}}{2x}\right)\right] \frac{x^{(p)}_{th} e^{-x}}{x^2}. \qquad (17)$$

Applying the asymptotic form of the modified Bessel function in Eq. (12) for the frequency range $n < n^{(p)}_{th}$ gives

$$f^{(p)}(x) \simeq f^{(p)}_0(x) = \frac{2\pi}{x} \exp\left(-\frac{x^{(p)}_{th}}{x} - x\right) \quad (x < x^{(p)}_{th}). \qquad (18)$$

Hence $I(p)$ in Eq. (16) is

$$I(p) \simeq I_0(p) + \Delta I(p);$$
$$I_0(p) = \frac{3g_p}{4} \int_0^\infty f^{(p)}_0(x)\, dx = 3\pi g_p K_0\left(2\sqrt{x^{(p)}_{th}}\right),$$
$$\Delta I(p) = \frac{3g_p}{4} \int_{x^{(p)}_{th}}^\infty [f^{(p)}(x) - f^{(p)}_0(x)]\, dx. \qquad (19)$$

The relative error of estimating $I(p)$ using $I_0(p)$ is plotted in Fig. 1 as a function of $x^{(p)}_{th}$. The plot shows that when $x^{(p)}_{th} \geq 1$, $\Delta I(p)/I_0(p)$ is negligibly small, and in this limit $I(p) \simeq I_0(p)$. This circumstance arises because, in the integrand of $\Delta I(p)$, the error introduced by using the asymptotic form of $K_\mu(z)$ is suppressed by a factor e^{-x} for $x \geq x^{(p)}_{th} > 1$. Moreover, when $x^{(p)}_{th} > 1$, we can use Eq. (12) to express $K_0\left(2\sqrt{x_{th}^{(p)}}\right)$ of Eq. (19) in its asymptotic form, after which we find $I_0(p) \propto p^{1/4}\exp[-2p^{3/2}(n_{th}/n_c)]$. This expression shows clearly that, when $n_{th} > n_c$, $I_0(p)$ rapidly decreases with increasing p, so in this strong-shielding limit, the CSR power

$$P_{coh} \simeq \frac{4N^2e^2c}{3\pi\rho h}\sum_{p=1,3}^{\infty} I_0(p) \quad (n_{th} > n_c) \qquad (20)$$

is dominated by $p = 1$ mode, and its ratio to the free-space steady-state CSR power in Eq. (10) gives

$$P_{coh}/P_{coh}^{(0)} \simeq C_0(n_{th}/n_c)^{5/6}\exp(-2n_{th}/n_c), \quad (n_{th} > n_c) \qquad (21)$$

with $C_0 = 4\sqrt{3\pi}/2^{2/3}[\Gamma(2/3)]^2 \simeq 4.2$. This modified result for the shielded CSR power in the parameter regime $n_{th} > n_c$ differs markedly with the result in Eq. (11).

To emphasize the importance of our findings, we compare the analytic results given in Eqs. (20) and (21) with the previous result in Eq. (11) by way of Figs. 2 and 3. The circular dots in Fig. 2 are obtained numerically from Eqs. (11) and (12) of Ref. [6] for the steady-state case, a result that derives from application of the image charge method. The crosses are obtained by direct summation of power harmonics in Eq. (6), in which the upper limit of the sum is chosen empirically by monitoring convergence of the result. It is clear that, for $n_{th} > n_c$, our simple result in Eq. (20) for $p = 1$ agrees well with exact calculations using the image charge method and superposition of power harmonics. It is also clear that a large number of p modes are needed only for the weak-shielding case when $n_{th}/n_c \ll 1$. An alternative way to view the results is provided in Fig. 3, in which the solid curve denotes the Kheifets-Zotter result, and the dashed curve denotes the result of Eq. (21) which is accurate for nearly the full range of values along the abscissa. Fig. 3 is to be compared with Fig. 9 in Ref. [3], in which our dashed curve is replaced by the exact results from the image-charge method. There is no discernible difference between the two figures. These comparisons show the validity of using Eq. (21) to describe $P_{coh}/P_{coh}^{(0)}$ when $n_{th} \geq n_c$, and they also underscore the validity of the image-charge method.

4 DISCUSSION

According to Ref. [5], for the $p = 1$ mode, the radiation intensity in the nth harmonic falls off exponentially as n decreases from n_{th} to zero. This is seen in Eqs. (4), (5) and (12) for $p = 1$, where $K_\mu^2(g_1^3/3n^2) \propto e^{-n_{th}^2/n^2}$ for $n \leq n_{th}$. If $n_{th} \ll n_c$, then the contribution of the radiation intensity in the range $0 < n \leq n_{th}$ is negligible. However, for $n_{th} > n_c$, all the bunch frequencies lie inside the range $0 < n \leq n_{th}$, and thus those harmonics are the main contributor to P_{coh}. Discarding these harmonics results in a potentially considerable underestimation of the radiated power.

5 REFERENCES

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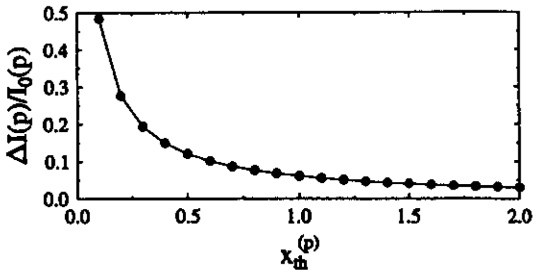

Figure 1: Relative error in using $I_0(p)$ to estimate $I(p)$ as a function of $x_{th}^{(p)}$, as obtained using Eq. (19).

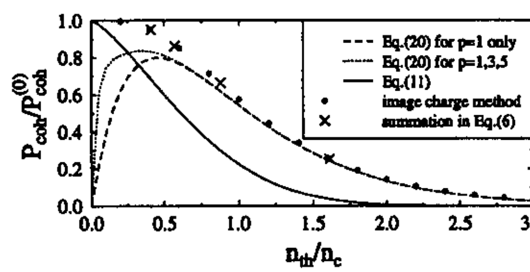

Figure 2: Comparison of P_{coh} in Eqs. (20) and (21) with results from the image-charge method and other approaches. Here n_{th}/n_c is varied by changing h with fixed values of ρ (= 1 m) and σ (=1 mm.)

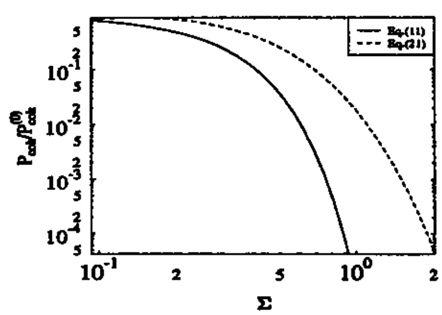

Figure 3: Plots of $P_{coh}/P_{coh}^{(0)}$ versus $\Sigma = \sigma/(2\rho\Delta^{3/2})$ with $\Delta \equiv h/(2\rho)$. This should be compared with Fig. 9 of Ref. [3].