

Normalization of the covariant three-body bound state vertex function

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(June 6, 2000)

Abstract

The normalization condition for the relativistic three nucleon Bethe-Salpeter and Gross bound state vertex functions is derived, for the first time, directly from the three body wave equations. It is also shown that the relativistic normalization condition for the two body Gross bound state vertex function is identical to the requirement that the bound state charge be conserved, proving that charge is automatically conserved by this equation.

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I. INTRODUCTION

The use of vertex functions derived from the covariant Bethe-Salpeter (BS) equation [1,2] or the covariant spectator (or Gross) equation [3,4] to calculate matrix elements involving bound states requires that the normalization of these vertex functions be determined. Although this may seem like a trivial problem, the complicated forms of these equations, along with the generally nonlocal nature of their interaction kernels makes it more than trivial. In fact we know of only one derivation of the normalization condition for the three-body BS bound state vertex functions [5], and this derivation does not seem to show that the normalization condition follows simply from application of the wave equation to the description of scattering in the region of the bound state, and does not obtain the normalization condition for the Faddeev subvertex functions. The three body spectator equations have only recently been derived in their final form [6] and this paper includes the first derivation of the normalization condition of the three body spectator vertex functions. Normalization conditions for the two-body BS and Gross equations have been previously obtained by many people (see, e.g., [7–9]), but this paper includes a first demonstration that charge is automatically conserved by the spectator equations.

Following this introductory discussion the paper begins, in Sec. II, with a derivation of the normalization condition for both the two-body BS and Gross vertex functions. These results, which are not new, are presented in order to introduce the techniques which will be used later in the derivation of the three-body conditions. At the conclusion of this section we prove, for the first time, that the normalization of the two-body Gross vertex function is identical to the requirement that the charge of the bound state, as defined by the charge operator introduced by Gross and Riska [10], is equal to the sum of the charges of its constituents. This shows that charge is automatically conserved by the spectator theory.

The third section contains the new derivations of the normalization conditions for the three-body Bethe-Salpeter and Gross vertex functions [4], which closely follow the approach used in the two-body case. A discussion of the Feynman rules used in this paper and the conventions used in symmetrizing the scattering matrices is presented in the Appendix.

II. NORMALIZATION OF RELATIVISTIC TWO-BODY VERTEX FUNCTIONS

The derivations of the normalization condition for two and three-body vertex functions are very similar. In this section we start with the simpler two-body case, and then generalize the derivation to the three body case in Sec. III.

A. Normalization of the BS two-body vertex function

The two-body Bethe-Salpeter equation for the scattering matrix \mathcal{M} is represented by the Feynman diagrams in Fig. 1. Using the Feynman rules described in the Appendix, this corresponds to

$$\mathcal{M} = V - VG_{BS}\mathcal{M} \tag{2.1}$$

$$= V - \mathcal{M}G_{BS}V, \tag{2.2}$$

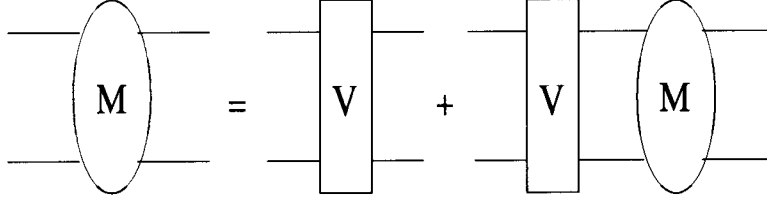


FIG. 1. Diagrammatic representation of the two-body BS equation for the scattering matrix.

where V is the two-body interaction kernel and G_{BS} is the free two-body propagator. In terms of the one body propagator G_i defined in Eq. (2.30) below, the BS propagator is

$$G_{BS} \equiv -i G_1 G_2. \quad (2.3)$$

From Eq. (2.2) we have

$$V = \mathcal{M} + \mathcal{M} G_{BS} V, \quad (2.4)$$

and substituting this equation into Eq. (2.1) gives the following nonlinear equation for \mathcal{M}

$$\mathcal{M} = V - \mathcal{M} G_{BS} \mathcal{M} - \mathcal{M} G_{BS} V G_{BS} \mathcal{M}. \quad (2.5)$$

If the two-body system has a bound state at $P^2 = M^2$, where P is the total four-momentum, the scattering amplitude \mathcal{M} will have a pole at $P^2 = M^2$. Therefore, the two-body scattering amplitude can be written

$$\mathcal{M} = -\frac{|\Gamma\rangle\langle\Gamma|}{M^2 - P^2} + \mathcal{R}, \quad (2.6)$$

where $|\Gamma\rangle$ is the Bethe-Salpeter vertex function and $\langle\Gamma|$ is the representation of $|\Gamma\rangle$ in the dual space, including Dirac conjugation. The first term is the contribution from the bound state propagator, with possible spin degrees of freedom associated with the propagation of the bound state suppressed, and \mathcal{R} is finite at $P^2 = M^2$. Substituting this form (2.6) into the linear form (2.1) of the equation for \mathcal{M} , and equating residues at $P^2 = M^2$ gives the **wave equation** for the vertex function $|\Gamma\rangle$

$$|\Gamma\rangle = -V G_{BS} |\Gamma\rangle. \quad (2.7)$$

Substituting (2.6) into the nonlinear form of the scattering equation (2.5), and keeping all terms which are singular near the bound state pole at $P^2 = M^2$, gives the following relation

$$|\Gamma\rangle\langle\Gamma| = \lim_{P^2 \rightarrow M^2} \left\{ |\Gamma\rangle \left[\frac{\langle\Gamma| G_{BS} (1 + V G_{BS}) |\Gamma\rangle}{M^2 - P^2} \right] \langle\Gamma| \right. \\ \left. - \mathcal{R} G_{BS} (1 + V G_{BS}) |\Gamma\rangle\langle\Gamma| - |\Gamma\rangle\langle\Gamma| (1 + G_{BS} V) G_{BS} \mathcal{R} \right\}. \quad (2.8)$$

Near the bound state pole at $P^2 = M^2$ the BS equation (2.7) ensures that the last two terms involving the remainder \mathcal{R} vanish. Therefore, as $P^2 \rightarrow M^2$, Eq. (2.8) reduces to

$$1 = \lim_{P^2 \rightarrow M^2} \frac{\langle \Gamma | G_{BS} (1 + VG_{BS}) | \Gamma \rangle}{M^2 - P^2}. \quad (2.9)$$

Since both the numerator and the denominator vanish in this limit, we expand the numerator around $P^2 = M^2$, giving

$$1 = -\langle \Gamma | \frac{\partial G_{BS}}{\partial P^2} | \Gamma \rangle - \langle \Gamma | \frac{\partial}{\partial P^2} (G_{BS} V G_{BS}) | \Gamma \rangle. \quad (2.10)$$

(The BS equation ensures that any terms proportional to $\partial|\Gamma\rangle/\partial P^2$ will also vanish.) Distributing the derivative in the second term over the product $G_{BS} V G_{BS}$ and using the bound-state BS equation (2.7) reduces the normalization condition to

$$1 = \langle \Gamma | G'_{BS} | \Gamma \rangle - \langle \Gamma | G_{BS} V' G_{BS} | \Gamma \rangle. \quad (2.11)$$

where

$$G'_{BS} = \frac{\partial G_{BS}}{\partial P^2} = \frac{P^\mu}{2P^2} \frac{\partial G_{BS}}{\partial P^\mu}, \quad V' = \frac{\partial V}{\partial P^2} = \frac{P^\mu}{2P^2} \frac{\partial V}{\partial P^\mu}, \quad (2.12)$$

and all derivatives are taken at the bound state pole $P_0 = E(\vec{P}) = (\vec{P}^2 + M^2)^{1/2}$.

For identical particles, the symmetrized scattering matrix, which we will denote by M , must satisfy

$$\mathcal{P}_{12} M = M \mathcal{P}_{12} = \zeta M \quad (2.13)$$

where \mathcal{P}_{12} is the permutation operator that exchanges all labels for particles 1 and 2, and $\zeta = 1$ for bosons and $\zeta = -1$ for fermions. This can be achieved by symmetrizing the unsymmetrized scattering matrix \mathcal{M} according to

$$M = \mathcal{A}_2 \mathcal{M} = \mathcal{A}_2 \mathcal{M} \mathcal{A}_2, \quad (2.14)$$

where

$$\mathcal{A}_2 = \frac{1}{2} (1 + \zeta \mathcal{P}_{12}) \quad (2.15)$$

is the symmetrization operator and the last relation follows from $\mathcal{A}_2 \mathcal{M} = \mathcal{M} \mathcal{A}_2$ and $\mathcal{A}_2^2 = \mathcal{A}_2$. Using the Bethe-Salpeter equation for the scattering matrix yields

$$\begin{aligned} M &= \mathcal{A}_2 \mathcal{M} = \mathcal{A}_2 (V - V G_{BS} \mathcal{M}) \\ &= \mathcal{A}_2 V - \mathcal{A}_2 V G_{BS} \mathcal{A}_2 \mathcal{M} \\ &= \mathcal{A}_2 V - \mathcal{A}_2 V G_{BS} M \end{aligned} \quad (2.16)$$

where the identities $\mathcal{A}_2 V = \mathcal{A}_2 V \mathcal{A}_2$ and $\mathcal{A}_2 G_{BS} = G_{BS} \mathcal{A}_2$ have been used in rewriting the expression. Defining the symmetrized kernel illustrated diagrammatically in Fig. 2

$$\bar{V} = \mathcal{A}_2 V = \frac{1}{2} (1 + \zeta \mathcal{P}_{12}) V, \quad (2.17)$$

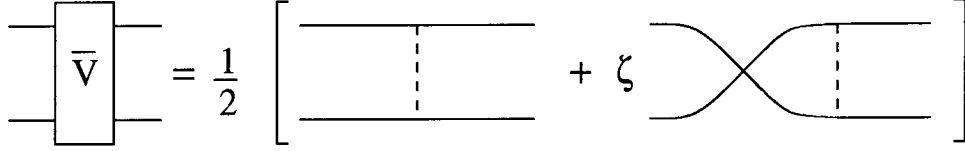


FIG. 2. Diagrammatic representation of the symmetrized kernel.

gives a Bethe-Salpeter equation for the symmetrized scattering matrix with the same form as the Bethe-Salpeter equation for nonidentical particles

$$M = \bar{V} - \bar{V}G_{BS}M. \quad (2.18)$$

Since the structure of this equation is the same as in the case of identical particles, the derivation of the bound state equation and the normalization condition is the same as before with replacement of $V \rightarrow \bar{V}$

$$1 = {}_s\langle \Gamma | G'_{BS} | \Gamma \rangle_s - {}_s\langle \Gamma | G_{BS} \bar{V}' G_{BS} | \Gamma \rangle_s, \quad (2.19)$$

where

$$|\Gamma\rangle_s = \mathcal{A}_2 |\Gamma\rangle \quad (2.20)$$

is the symmetrized vertex function.

Note that the symmetrized scattering matrix defined above is normalized differently from the symmetrized scattering matrices presented in most field theory texts, as described in the Appendix.

B. Normalization of the Gross two-body vertex functions

The Gross or spectator equation for distinguishable particles can be obtained from the corresponding Bethe-Salpeter equation by means of a simple prescription. For unequal mass particles with relatively long-range interactions, the heavier of the two particles is placed on its positive-energy mass shell in performing the energy loop integral over intermediate state four-momenta. Here, the heavier particle is assumed to be particle 1. The scattering matrix then becomes

$$\mathcal{M} = V - VQ_1G_2\mathcal{M} \quad (2.21)$$

where the projection operator $Q_i = Q_i^2$ places particle i on its positive-energy mass shell. Equation (2.21) does not represent a closed set of solvable equations. In order to obtain these it is necessary to also place particle 1 on shell in both the initial and final states leading to

$$\mathcal{M}_{11} = V_{11} - V_{11}G_2\mathcal{M}_{11} \quad (2.22)$$

where $V_{ii} = Q_i V Q_i$ and $\mathcal{M}_{ii} = Q_i \mathcal{M} Q_i$. This is illustrated in Fig. 3.

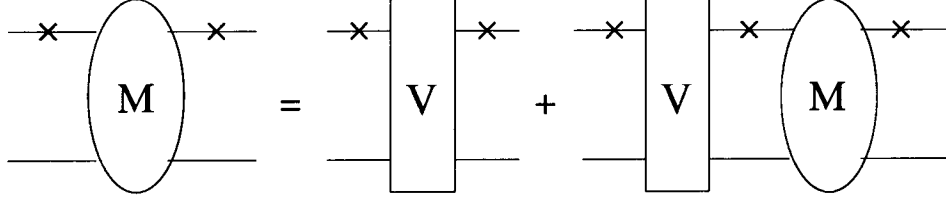


FIG. 3. Diagrammatic representation of the two-body Gross equation for the scattering matrix. The \times on the line for particle 1 indicates that it is on shell.

The situation for identical particles is somewhat more complicated than for the Bethe-Salpeter equation. Since $\mathcal{P}_{12}G_2\mathcal{P}_{12} = G_1$ and $\mathcal{P}_{12}\mathcal{Q}_1\mathcal{P}_{12} = \mathcal{Q}_2$, the scattering matrix can no longer be symmetrized by simply symmetrizing the final state. The solution to this problem is to use a symmetric version of the spectator equation

$$\mathcal{M} = V - \frac{1}{2} V (\mathcal{Q}_1 G_2 + \mathcal{Q}_2 G_1) \mathcal{M} \quad (2.23)$$

where the factor of $\frac{1}{2}$ is necessary to prevent double counting of the elastic unitary cut of the scattering matrix. The factor of $\frac{1}{2}$ emerges automatically if we think of obtaining Eq. (2.23) from the BS equation by doing the integration over the relative energy variable by averaging the results obtained from closing the contour in the upper half and lower half planes (and keeping the positive energy nucleon poles only). Eq. (2.23) can now be symmetrized in the same way as the Bethe-Salpeter equation, giving

$$\begin{aligned} M &= \bar{V} - \frac{1}{2} \bar{V} (\mathcal{Q}_1 G_2 + \mathcal{Q}_2 G_1) M \\ &= \bar{V} - \frac{1}{2} \bar{V} (\mathcal{Q}_1 G_2 + \mathcal{P}_{12} \mathcal{Q}_1 G_2 \mathcal{P}_{12}) M \\ &= \bar{V} - \frac{1}{2} \bar{V} \mathcal{Q}_1 G_2 (1 + \zeta \mathcal{P}_{12}) M \\ &= \bar{V} - \bar{V} \mathcal{Q}_1 G_2 M. \end{aligned} \quad (2.24)$$

Clearly, $\mathcal{Q}_1 G_2$ could be eliminated in favor of $\mathcal{Q}_2 G_1$ in exactly the same manner. Therefore, the M -matrix is independent of the choice of on-shell particle, and the resulting integral equation for the scattering matrix for (2.24) can be taken to be

$$M_{11} = \bar{V}_{11} - \bar{V}_{11} G_2 M_{11}, \quad (2.25)$$

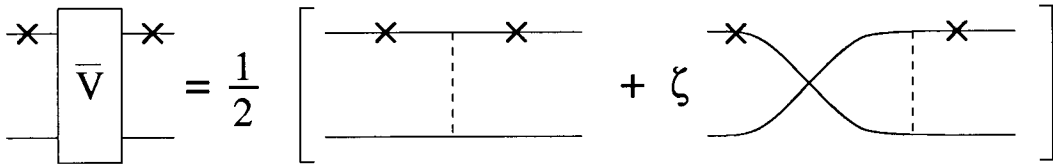


FIG. 4. Diagrammatic representation of the symmetrized kernel for Gross equation. As in Fig. 3, the \times indicates that the particle is on shell.

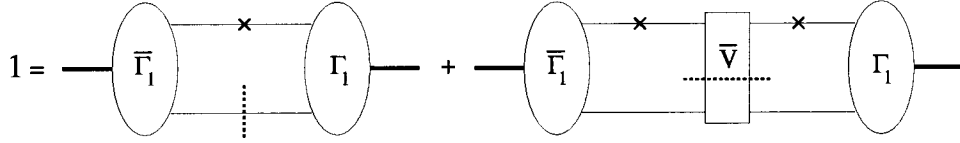


FIG. 5. Diagrammatic representation of the normalization condition for the two-body Gross vertex function. The dashed line represents the derivative $\partial/\partial P^2$, and the \times means that the particle is on shell.

where the symmetrized potential is shown in Fig. 4.

As for the Bethe-Salpeter equation, the existence of a bound state implies a pole in the scattering matrix,

$$M_{11} = -\frac{|\Gamma_1\rangle\langle\Gamma_1|}{M^2 - P^2} + \mathcal{R}, \quad (2.26)$$

where $|\Gamma_1\rangle$ is the bound state vertex function with particle 1 on shell. Repeating the development used in the preceding section gives the Gross equation for $|\Gamma_1\rangle$

$$|\Gamma_1\rangle = -\bar{V}_{11}G_2|\Gamma_1\rangle. \quad (2.27)$$

We will choose the momentum of the on-shell particle and the total momentum P as independent variables so that $\partial/\partial P_\mu$ refers to differentiation with respect to P_μ holding *the momentum of the on-shell particle constant*. Hence the on-shell projection operators \mathcal{Q} will be independent of P_μ , and the normalization condition becomes:

$$1 = \langle\Gamma_1|G'_2|\Gamma_1\rangle - \langle\Gamma_1|G_2\bar{V}'_{11}G_2|\Gamma_1\rangle. \quad (2.28)$$

This equation is represented diagrammatically in Fig. 5. Note that this result could also be obtained by applying the spectator prescription directly to the unsymmetrized version of the Bethe-Salpeter normalization condition (2.19). Clearly, a similar expression can be derived for Γ_2 .

C. Conservation of electromagnetic charge for the Gross equation

Since we have obtained the normalization condition (2.28) directly from the two-body equation, with no reference to the charge of the bound state, we now are in a position to *prove* that the charge of the bound state is conserved; ie. that the charge of the bound state is the sum of the charges of its two constituents, regardless of its structure.

The construction of the matrix elements for the two-body current for the Gross equation have been discussed by Gross and Riska [10] and applied to elastic electron scattering from deuteron in [9]. As argued in [9], the bound state matrix elements of the electromagnetic current in the simple case when the OBE kernel is constructed from the the exchange of *neutral* bosons is represented by the Feynman diagrams shown in Fig. 6. Notice that since in our spectator formulation only the first particle is being placed on-shell, the first diagram

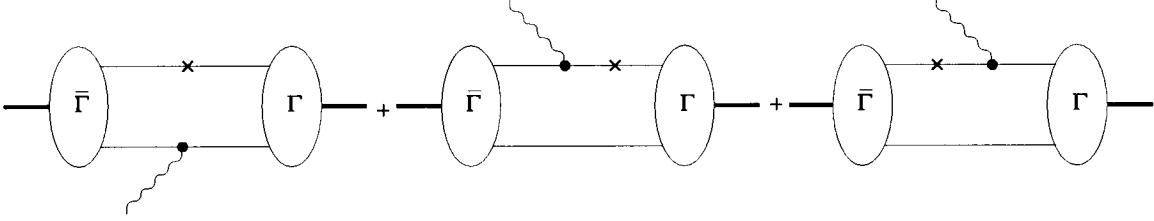


FIG. 6. Diagrams which contribute to the electromagnetic charge of the two body bound state.

in Fig. 6 with $1 \leftrightarrow 2$ does not appear. Here we show explicitly that the last two diagrams in Fig. 6 give a contribution in the limit $q \rightarrow 0$ equal to that of the first diagram as well as the derivative term of the potential as it appears in the normalization condition (2.28).

Our argument will require that we replace the operator form of the Gross equation (2.27) by its explicit representation in momentum space. We will use the notation

$$N = \begin{cases} 1 & \text{for bosons} \\ 2m & \text{for fermions} \end{cases} \quad (2.29)$$

and

$$\mathcal{Q}_i(\hat{p}) \rightarrow \Lambda_i(\hat{p}) = \begin{cases} 1 & \text{for bosons} \\ \frac{\not{\hat{p}} + m_i}{2m_i} & \text{for fermions} \end{cases} \quad G_i(p) = \begin{cases} \frac{1}{m_i^2 - p^2 - i\epsilon} & \text{for bosons} \\ \frac{1}{m_i - \not{p} - i\epsilon} & \text{for fermions,} \end{cases} \quad (2.30)$$

where Λ_i is what remains of the projection operator \mathcal{Q}_i after the integral over the relative energy has been carried out and the delta function which places particle 1 on shell has been removed, and \hat{p} denotes an on-shell four-momentum with $\hat{p}^2 = m^2$. The equation then becomes

$$\Gamma(\hat{p}_1; P) = - \int \frac{d^3 k_1 N}{(2\pi)^3 2E_{k_1}} \bar{V}(\hat{p}_1, \hat{k}_1; P) \Lambda_1(\hat{k}_1) G_2(P - \hat{k}_1) \Gamma(\hat{k}_1; P). \quad (2.31)$$

Note that the vertex functions are written as a function of the four-momentum of particle 1, \hat{p}_1 , and the total four-momentum, P . Multiplying both sides of the equation by the projection operator $\Lambda_1(\hat{p}_1)$ and using $\Lambda_1^2(\hat{k}_1) = \Lambda_1(\hat{k}_1)$ gives

$$\Gamma_1(\hat{p}_1; P) = - \int \frac{d^3 k_1 N}{(2\pi)^3 2E_{k_1}} \bar{V}_{11}(\hat{p}_1, \hat{k}_1; P) G_2(P - \hat{k}_1) \Gamma_1(\hat{k}_1; P), \quad (2.32)$$

where $\Lambda_1(\hat{p}_1) \bar{V}(\hat{p}_1, \hat{k}_1; P) \Lambda_1(\hat{k}_1) = \bar{V}_{11}(\hat{p}_1, \hat{k}_1; P)$ and $\Gamma_1(\hat{p}_1; P) = \Lambda_1(\hat{p}_1) \Gamma(\hat{p}_1; P)$.

Using the Feynman rules, and the notation used in Eq. (2.31) and (2.32), the diagrams in Fig. 6 yield

$$\begin{aligned}
\mathcal{J}^\mu(P', P) &= \int \frac{d^3 p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}_1; P') G_2(P' - \hat{p}_1) j_2^\mu(P' - \hat{p}_1, P - \hat{p}_1) G_2(P - \hat{p}_1) \Gamma_1(\hat{p}_1; P) \\
&+ \int \frac{d^3 p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}(\hat{p}_1 + q; P') G_1(\hat{p}_1 + q) j_1^\mu(\hat{p}_1 + q, \hat{p}_1) G_2(P - \hat{p}_1) \Gamma_1(\hat{p}_1; P) \\
&+ \int \frac{d^3 p'_1 N}{(2\pi)^3 2E_{p'_1}} \bar{\Gamma}_1(\hat{p}'_1; P') j_1^\mu(\hat{p}'_1, \hat{p}'_1 - q) G_1(\hat{p}'_1 - q) G_2(P' - \hat{p}'_1) \Gamma(\hat{p}'_1 - q; P), \quad (2.33)
\end{aligned}$$

where $j_i^\mu(p', p)$ is the renormalized e.m. current of i -th constituent particle. When taken between the on-shell states for $p' = p$, this current defines a physical charge e_i . The one-body currents satisfy the one-body Ward-Takahashi identity

$$q_\mu j_i^\mu(p'_i, p_i) = -e_i \left(\tilde{G}_i^{-1}(p') - \tilde{G}_i^{-1}(p) \right) \simeq -e_i q_\mu \frac{\partial}{\partial p_\mu} \tilde{G}_i^{-1}(p), \quad (2.34)$$

where \tilde{G} is a full one-particle propagator and the last form of the identity holds near the point $q_\mu \rightarrow 0$. Comparing the terms linear in q_μ we get from (2.34)

$$j_i^\mu(p, p) = -e_i \frac{\partial}{\partial p_\mu} \tilde{G}_i^{-1}(p). \quad (2.35)$$

Any purely transverse parts $j_{T,i}^\mu(p', p)$ of the off-shell one-body current, which satisfy $q_\mu j_{T,i}^\mu(p', p) = 0$ and cannot be determined from (2.34), are assumed to vanish in the limit $q \rightarrow 0$. Therefore, they do not affect the charge e_i . Next, note the identity

$$\tilde{G}_i(p) j_i^\mu(p, p) \tilde{G}_i(p) = -e_i \tilde{G}_i(p) \left(\frac{\partial}{\partial p_\mu} \tilde{G}_i^{-1}(p) \right) \tilde{G}_i(p) = e_i \frac{\partial}{\partial p_\mu} \tilde{G}_i(p). \quad (2.36)$$

Thus, under the assumptions discussed above the insertion of the $q = 0$ photon into the line of the constituent particle is unambiguously determined by its physical charge and derivative of its propagator. In this paper we always approximate \tilde{G} by (2.30) with a physical mass m_i ; our arguments are however valid also for the generalized $\tilde{G}_i(p) = h^2(p^2)/(m_i - \not{p})$ used in a dynamical model of Ref. [8].

Now, the charge of the bound system is found by taking the limit of (2.33) $q \rightarrow 0$, or $P' \rightarrow P$, since contributions from any higher multipoles, such as magnetic dipole or charge quadrupole must vanish for $q = 0$. In the limit considered the propagators for particle 1 in the last two terms, $G_1(\hat{p}_1 + q)$ and $G_1(\hat{p}'_1 - q)$, become singular. However, it is easy to see that these singularities cancel to give a finite result.

Before we proceed with the evaluation of the $q \rightarrow 0$ limit of Eq. (2.33), it is instructive to describe this cancellation of singularities in very general terms. To this end we note that the *last two terms* of Eq. (2.33) can be identically represented by the four dimensional integral

$$\begin{aligned}
\mathcal{J}^\mu(P', P)|_{\text{last 2 terms}} &= -i \int_C \frac{d^4 p_1}{(2\pi)^4} \bar{\Gamma}(p_1 + q; P') G_1(p_1 + q) j_1^\mu(p_1 + q, p_1) \\
&\times G_1(p_1) G_2(P - p_1) \Gamma(p_1; P), \quad (2.37)
\end{aligned}$$

where the integration over p_{10} is done along the contour C in the complex p_{10} plane, shown in Fig. 7. This contour surrounds the positive energy poles of the propagators $G_1(p_1 + q)$ (at

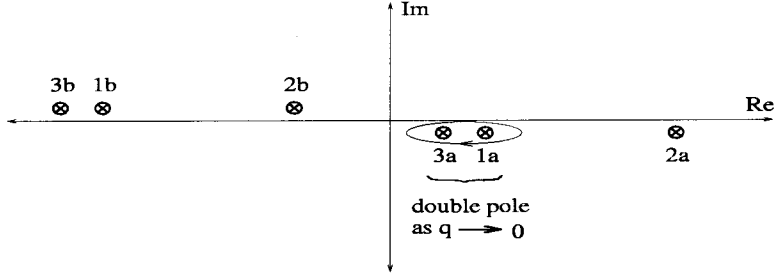


FIG. 7. Location in the p_{10} complex plane of the 6 poles from the three nucleon propagators in Eq. (2.34). The last two terms in Eq. (2.33) emerge if the integral of p_{10} over the contour C enclosing the poles 3a and 1a is evaluated using the residue theorem.

3a) and $G_1(p_1)$ (at 1a), and evaluating the integral by the residue theorem gives *only* the last two terms in Eq. (2.33), where for notational simplicity in the third term the integral over d^3p_1 is replaced by the integral over $d^3p'_1$ (where $\mathbf{p}'_1 = \mathbf{p}_1 + \mathbf{q}$). It has been shown [11] that for elastic reactions the cuts of $\Gamma(p_1; P)$ never overlap with poles 3a and 1a of Fig. 7, this makes the representation (2.37) possible and unambiguous for all values of momentum transfer q . It is now clear that as $q \rightarrow 0$ the poles at 3a and 1a coalesce into a double pole, giving a finite result. Calculation of the residue of the double pole requires a calculation of the derivative of the rest of the integrand, which also explains the appearance of derivatives in the final result below.

We now return to the reduction of Eq. (2.33). Substituting (2.36) into the first term of Eq. (2.33) gives

$$\mathcal{J}^\mu(P, P)|_{\text{first term}} = e_2 \int \frac{d^3p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}_1; P') \frac{\partial}{\partial P_\mu} G_2(P - \hat{p}_1) \Gamma_1(\hat{p}_1; P), \quad (2.38)$$

where $\partial/\partial(P - \hat{p}_1)_\mu$ was replaced by $\partial/\partial P_\mu$, which is possible only because \hat{p}_1 is an on-shell four-momentum, so that all four of its components are independent of P . We will return to this expression after we have reduced the last two terms.

The last two (singular) terms in Eq. (2.33) can be reduced by expressing the vertex function with both particles off-shell in terms of the vertex function with particle 1 on-shell by using the equation

$$\Gamma(p_1; P) = - \int \frac{d^3k_1 N}{(2\pi)^3 2E_{k_1}} \bar{V}(p_1, \hat{k}_1; P) G_2(P - \hat{k}_1) \Gamma_1(\hat{k}_1; P), \quad (2.39)$$

where the kernel $\bar{V}(p_1, \hat{k}_1; P)$ now connects an incoming channel with particle 1 on-shell to an outgoing channel with *both* particles off-shell and is obtained from $\bar{V}(\hat{p}_1, \hat{k}_1; P)$ by replacing the four momentum $\hat{p}_1 = (E_p, \mathbf{p})$ by the off-shell four-momentum $(P_0 - E_{P-p}, \mathbf{p})$. [Eq. (2.39) is obtained from the original Eq. (2.21) by going to the bound state pole in the usual way.] Using this equation to iterate each of the off-shell vertex functions once yields

$$\begin{aligned} \mathcal{J}^\mu(P', P)|_{\text{last 2 terms}} = & \int \frac{d^3p'_1 N}{(2\pi)^3 2E_{p'_1}} \int \frac{d^3p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}'_1; P') G_2(P' - \hat{p}'_1) j_{\text{eff}}^\mu(\hat{p}'_1, P'; \hat{p}_1, P) \\ & \times G_2(P - \hat{p}_1) \Gamma_1(\hat{p}_1; P) \end{aligned} \quad (2.40)$$

where

$$\begin{aligned}
j_{\text{eff}}^\mu(\hat{p}'_1, P'; \hat{p}_1, P) &= \int \frac{d^3 k_1 N}{(2\pi)^3 2E_{k_1}} \Lambda_1(\hat{p}'_1) \left[\bar{V}(\hat{p}'_1, \hat{k}_1 + q; P') G_1(\hat{k}_1 + q) \right. \\
&\quad \times j_1^\mu(\hat{k}_1 + q, \hat{k}_1) \Lambda_1(\hat{k}_1) G_2(P - \hat{k}_1) \bar{V}(\hat{k}_1, \hat{p}_1; P) + \\
&\quad \bar{V}(\hat{p}'_1, \hat{k}_1; P') G_2(P' - \hat{k}_1) \Lambda_1(\hat{k}_1) j_1^\mu(\hat{k}_1, \hat{k}_1 - q) \\
&\quad \left. \times G_1(\hat{k}_1 - q) \bar{V}(\hat{k}_1 - q, \hat{p}_1; P) \right] \Lambda_1(\hat{p}_1), \tag{2.41}
\end{aligned}$$

Using the WT identity (2.34) for finite q , and recalling that $G_1^{-1}(\hat{k}_1) \Lambda_1(\hat{k}_1) = 0$, gives for the divergence of the effective current

$$\begin{aligned}
q_\mu j_{\text{eff}}^\mu(\hat{p}'_1, P'; \hat{p}_1, P) &= -e_1 \Lambda_1(\hat{p}'_1) \int \frac{d^3 k_1 N}{(2\pi)^3 2E_{k_1}} \\
&\quad \times \left[\bar{V}(\hat{p}'_1, \hat{k}_1 + q; P') G_2(P - \hat{k}_1) \Lambda_1(\hat{k}_1) \bar{V}(\hat{k}_1, \hat{p}_1; P) \right. \\
&\quad \left. - \bar{V}(\hat{p}'_1, \hat{k}_1; P') \Lambda_1(\hat{k}_1) G_2(P' - \hat{k}_1) \bar{V}(\hat{k}_1 - q, \hat{p}_1; P) \right] \Lambda_1(\hat{p}_1). \tag{2.42}
\end{aligned}$$

Note that this step removes the propagators $G_1(\hat{k}_1 + q)$ and $G_1(\hat{k}_1 - q)$ which are singular in the $q \rightarrow 0$ limit. The purely transverse contributions to the effective current (2.41) arise only from the transverse parts of the one-body current and so, by assumption, vanish as $q \rightarrow 0$. Therefore, we can use the divergence of the effective current to extract its finite part for $P' = P$. As before we expand (2.42) about the point $q = 0$ and equate terms linear in q_μ to obtain

$$\begin{aligned}
j_{\text{eff}}^\mu(\hat{p}'_1, P; \hat{p}_1, P) &= -e_1 \int \frac{d^3 k_1 N}{(2\pi)^3 2E_{k_1}} \Lambda_1(\hat{p}'_1) \\
&\quad \times \left[\left(\frac{\partial}{\partial \hat{k}_{1\mu}} \bar{V}(\hat{p}'_1, \hat{k}_1; P) \right) G_2(P - \hat{k}_1) \Lambda_1(\hat{k}_1) \bar{V}(\hat{k}_1, \hat{p}_1; P) \right. \\
&\quad - \bar{V}(\hat{p}'_1, \hat{k}_1; P) \left(\frac{\partial}{\partial P_\mu} G_2(P - \hat{k}_1) \right) \Lambda_1(\hat{k}_1) \bar{V}(\hat{k}_1, \hat{p}_1; P) \\
&\quad \left. + \bar{V}(\hat{p}'_1, \hat{k}_1; P) G_2(P - \hat{k}_1) \Lambda_1(\hat{k}_1) \left(\frac{\partial}{\partial \hat{k}_{1\mu}} \bar{V}_{11}(\hat{k}_1, \hat{p}_1; P) \right) \right] \Lambda_1(\hat{p}_1). \tag{2.43}
\end{aligned}$$

Substituting this into Eq. (2.40), and using $\Lambda_1^2 = \Lambda_1$ and the spectator equation (2.32) for the bound state gives

$$\begin{aligned}
\mathcal{J}^\mu(P, P)|_{\text{terms}}^{\text{last 2}} &= e_1 \int \frac{d^3 p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}_1; P) \left(\frac{\partial}{\partial P_\mu} G_2(P - \hat{p}_1) \right) \Gamma_1(\hat{p}_1; P) \\
&\quad + e_1 \int \int \frac{d^3 p'_1 d^3 p_1 N^2}{(2\pi)^6 4E_{p'_1} E_{p_1}} \bar{\Gamma}_1(\hat{p}'_1; P) G_2(P - \hat{p}'_1) \left(\frac{\partial}{\partial \hat{p}'_{1\mu}} \bar{V}(\hat{p}'_1, \hat{p}_1; P) \right. \\
&\quad \left. + \frac{\partial}{\partial \hat{p}'_{1\mu}} \bar{V}(\hat{p}'_1, \hat{p}_1; P) \right) G_2(P - \hat{p}_1) \Gamma_1(\hat{p}_1; P). \tag{2.44}
\end{aligned}$$

Note that the first term is identical to Eq. (2.38) if $e_1 \rightarrow e_2$. To reduce the last two terms use the fact that the symmetrized one boson exchange (OBE) potential has the form

$$\bar{V}(\hat{p}'_1, \hat{p}_1; P) = \frac{1}{2} (1 + \zeta \mathcal{P}_{12}) V(\hat{p}'_1, \hat{p}_1; P) = \frac{1}{2} [V_d(\hat{p}_1 - \hat{p}'_1) + \zeta V_e(\hat{p}_1 + \hat{p}'_1 - P)], \quad (2.45)$$

where V_d is the *direct* term and V_e the *exchange* term as defined in Ref. [8]. Therefore only the derivatives of the exchange term contribute to the charge operator, and

$$\begin{aligned} \left(\frac{\partial}{\partial \hat{p}_{1\mu}} + \frac{\partial}{\partial \hat{p}'_{1\mu}} \right) \bar{V}(\hat{p}'_1, \hat{p}_1; P) &= \frac{1}{2} \zeta \left(\frac{\partial}{\partial \hat{p}_{1\mu}} + \frac{\partial}{\partial \hat{p}'_{1\mu}} \right) V_e(\hat{p}_1 + \hat{p}'_1 - P) \\ &= -\zeta \frac{\partial}{\partial P_\mu} V_e(\hat{p}_1 + \hat{p}'_1 - P) = -2 \frac{\partial}{\partial P_\mu} \bar{V}(\hat{p}'_1, \hat{p}_1; P), \end{aligned} \quad (2.46)$$

where we again used the fact that \hat{p}_1 and \hat{p}'_1 are both independent of P . Combining all of these results together gives our final result for the charge operator of the bound state

$$\begin{aligned} \mathcal{J}^\mu(P, P) &= (e_1 + e_2) \int \frac{d^3 p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}_1; P) \frac{\partial G_2(\hat{p}_1; P)}{\partial P_\mu} \Gamma_1(\hat{p}_1; P) \\ &\quad - 2e_1 \int \int \frac{d^3 p'_1 d^3 p_1 N^2}{(2\pi)^6 4E_{p'_1} E_{p_1}} \bar{\Gamma}_1(\hat{p}'_1; P) G_2(\hat{p}'_1; P) \frac{\partial \bar{V}_{11}(\hat{p}'_1, \hat{p}_1; P)}{\partial P_\mu} G_2(\hat{p}_1; P) \Gamma_1(\hat{p}_1; P) \Big] \\ &= e_B 2P^\mu. \end{aligned} \quad (2.47)$$

where we used $\Lambda_1^2 = \Lambda_1$ and $\Lambda_1(\hat{p}'_1) [\partial \bar{V}(\hat{p}'_1, \hat{p}_1; P) / \partial P_\mu] \Lambda_1(\hat{p}_1) = \partial \bar{V}_{11}(\hat{p}'_1, \hat{p}_1; P) / \partial P_\mu$. In the last step we also assumed, for definiteness, that the bound state has spin zero and total charge e_B . Multiplying the equation by P_μ and using the identity

$$P_\mu \left(\frac{\partial \mathcal{O}}{\partial P_\mu} \right) = 2P^2 \left(\frac{\partial \mathcal{O}}{\partial P^2} \right), \quad (2.48)$$

gives the relation

$$\begin{aligned} e_B &= (e_1 + e_2) \int \frac{d^3 p_1 N}{(2\pi)^3 2E_{p_1}} \bar{\Gamma}_1(\hat{p}_1; P) G'_2(\hat{p}_1; P) \Gamma_1(\hat{p}_1; P) \\ &\quad - 2e_1 \int \int \frac{d^3 p'_1 d^3 p_1 N^2}{(2\pi)^6 4E_{p'_1} E_{p_1}} \bar{\Gamma}_1(\hat{p}'_1; P) G_2(\hat{p}'_1; P) \bar{V}'_{11}(\hat{p}'_1, \hat{p}_1; P) G_2(\hat{p}_1; P) \Gamma_1(\hat{p}_1; P) \Big]. \end{aligned} \quad (2.49)$$

We now distinguish two cases. If the two particles are not identical, the OBE kernel does not include an exchange term and the \bar{V}'_{11} term vanishes [recall Eq. (2.46)]. In this case the normalization condition gives $e_B = e_1 + e_2$. If the particles are identical, the \bar{V}'_{11} term will not vanish, but the charges will be equal, so $e_1 + e_2 = 2e_1 = e_B$ and we obtain the same result. For either identical or distinguishable particles, the *normalization condition ensures the conservation of charge*.

This proof can be generalized to kernels which include multiple meson exchange. In this case the additional energy dependence contained in the kernel will lead to new contributions to the V' term in the normalization condition, but these new terms will be reproduced

exactly by the $q \rightarrow 0$ limit of additional Feynman diagrams containing interaction currents also required by the multiple meson exchange kernel, and the charge will still be conserved.

The charge conservation and its relation to the normalization condition can be conveniently discussed in terms of N-particle Ward-Takahashi identities [12]. We will present these identities together with the general e.m. currents for two- and three-body systems for the spectator formalism in a forthcoming paper [13].

III. NORMALIZATION OF RELATIVISTIC THREE-BODY VERTEX FUNCTIONS

In this final section we find the normalization condition for relativistic three-body vertex functions. The derivation for the three-body Bethe-Salpeter equation is formally identical to that given above for the two-body case. We briefly repeat it just to set a stage for discussion of more complicated spectator case. In particular, we discuss in some detail the symmetrization necessary for the identical particles and at the end we express our normalization condition in terms of the symmetrized Faddeev subamplitudes.

A. Normalization of three-body Bethe-Salpeter vertex functions

1. Distinguishable Particles

The three-body Bethe-Salpeter equation for distinguishable particles can be written

$$\mathcal{T} = \mathcal{V} - \mathcal{V}G_{BS}^0\mathcal{T} \quad (3.1)$$

$$= \mathcal{V} - \mathcal{T}G_{BS}^0\mathcal{V}, \quad (3.2)$$

where $G_{BS}^0 = -G_1G_2G_3$ is the three body BS propagator, and

$$\mathcal{V} = i \sum_i V^i G_i^{-1} = \sum_i \mathcal{V}^i \quad (3.3)$$

is the sum of all products of two body potentials V^i multiplied by the inverse of the propagator $-iG_i$ of the third, non-interacting spectator, and we are assuming that there are no relativistic three body forces (ie. diagrams which are three-body irreducible). In these equations and those which follow we have adopted the odd-man-out or spectator notation. If i, j and k represent some permutation of 1, 2 and 3, the two-body potential V^i represents the interaction of particles j and k .

These equations look deceptively simple, but the singularities in \mathcal{V} make them difficult to solve, and they are usually reduced to Faddeev form by introducing the subamplitudes $\mathcal{T}^{ii'}$ in which particle i' is the initial spectator and particle i the final spectator. Since these objects represent components of a summed Feynman perturbation expansion of the scattering problem, the terms “initial” and “final” refer to the topological form of the corresponding Feynman diagrams, not to temporal priority. Dropping the convention of summation over repeated indices, so that *all sums will be written explicitly*, the initial equations for these subamplitudes follow immediately from (3.1) and (3.2):

$$\mathcal{T}^{ii'} = i \delta_{ii'} V^i G_i^{-1} - V^i G_{BS}^i \sum_{\ell} \mathcal{T}^{\ell i'} \quad (3.4)$$

$$= i \delta_{ii'} V^i G_i^{-1} - \sum_{\ell} \mathcal{T}^{i\ell} G_{BS}^{i'} V^{i'}, \quad (3.5)$$

where the singularities have been canceled using $i G_i^{-1} G_{BS}^0 = -i G_j G_k \equiv G_{BS}^i$. Note that

$$\mathcal{V}^i G_{BS}^0 = V^i G_{BS}^i. \quad (3.6)$$

We will adopt the convention that the index ℓ can take on any value, and that i' , j' , and k' are all different indices describing initial state particles (just as i , j , and k describe final state particles). Note that

$$\mathcal{T} = \sum_{ii'} \mathcal{T}^{ii'} \quad (3.7)$$

and that Eq. (3.1) follows by summing (3.4) over i and i' , and that Eq. (3.2) follows by summing (3.5) over i and i' . Convergence of these equations can be further improved by moving the $\mathcal{T}^{ii'}$ from the rhs to the lhs and introducing the two-body scattering amplitude

$$\mathcal{M}^i = \left(1 + V^i G_{BS}^i\right)^{-1} V^i = V^i \left(1 + V^i G_{BS}^i\right)^{-1}. \quad (3.8)$$

This gives the famous Faddeev equations for the subamplitudes

$$\mathcal{T}^{ii'} = i \delta_{ii'} \mathcal{M}^i G_i^{-1} - \mathcal{M}^i G_{BS}^i \sum_{\ell \neq i} \mathcal{T}^{\ell i'} \quad (3.9)$$

$$= i \delta_{ii'} \mathcal{M}^i G_i^{-1} - \sum_{\ell \neq i'} \mathcal{T}^{i\ell} G_{BS}^{i'} \mathcal{M}^{i'}. \quad (3.10)$$

The Feynman diagrams representing Eq. (3.9) are shown in Fig. 8. In these figures the solid dot denotes the spectator.

Since the logic behind the resummation of the multiple scattering series in terms of Faddeev amplitudes is the same as in the nonrelativistic three-body problem, it is not surprising that the equations have a form similar to the nonrelativistic Faddeev equations [14]. It is important to remember, however, that in this case the summation is done in terms of Feynman perturbative expansion rather than the time-ordered perturbation theory of the nonrelativistic case. **As a result global three-body propagators are replaced by products of one-body Feynman propagators and an inverse propagator must be included in the driving term to ensure that no propagator appears for external legs of the three-body T matrix.** Similarly, due to similarities in the underlying logic, many of the expressions derived below have nonrelativistic analogues of similar form. The connection between the relativistic formulation of the three-body problem and the usual nonrelativistic approach will be explored elsewhere in detail.

A bound state of the three-body system is associated with the occurrence of a pole at $P^2 = M^2$ in the three-body scattering amplitude \mathcal{T} (and in each of the subamplitudes $\mathcal{T}^{ii'}$), where P is the total momentum of the three-body system and M is the mass of the bound state. Therefore, \mathcal{T} and $\mathcal{T}^{ii'}$ can be written as the sum of a pole term and a regular part $R^{ii'}$

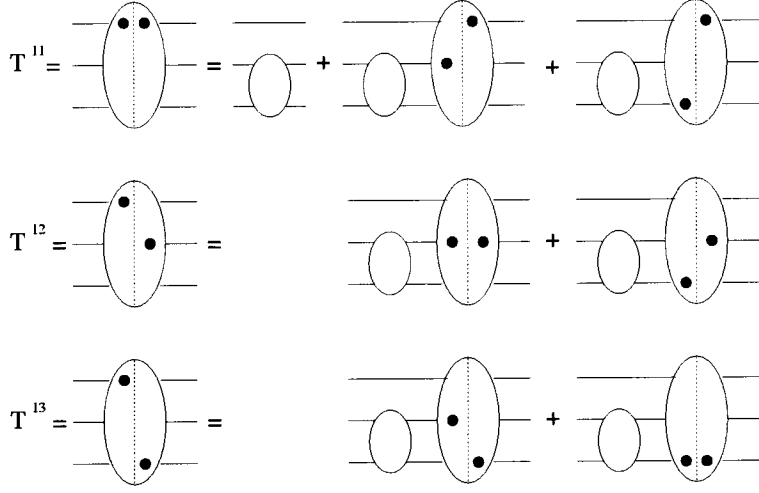


FIG. 8. Diagrammatic representation of the Faddeev equations for the amplitudes \mathcal{T}^{1i} . Note that the spectator is identified by the solid dot.

$$\mathcal{T}^{ii'} = -\frac{|\Gamma^i\rangle\langle\Gamma^{i'}|}{M^2 - P^2} + R^{ii'} \quad (3.11)$$

$$\mathcal{T} = -\frac{|\Gamma\rangle\langle\Gamma|}{M^2 - P^2} + \sum_{i'} R^{ii'}, \quad (3.12)$$

where $|\Gamma\rangle = \sum_i |\Gamma^i\rangle$. Inserting the ansatz (3.11) into the scattering equations and taking the limit $P^2 \rightarrow M^2$ gives the equation for the bound state vertex subamplitudes

$$|\Gamma^i\rangle = -\mathcal{M}^i G_{BS}^i \sum_{\ell \neq i} |\Gamma^\ell\rangle. \quad (3.13)$$

Similarly, using either Eq. (3.8) or starting with the equations for \mathcal{T} , we obtain

$$|\Gamma\rangle = -\sum_{\ell} V^\ell G_{BS}^\ell |\Gamma\rangle = -\mathcal{V} G_{BS}^0 |\Gamma\rangle. \quad (3.14)$$

The derivation of the three-body normalization condition is most easily obtained from the ansatz (3.12) and the original equations (3.1) and (3.2). Repeating the steps which lead to Eq. (2.11) we obtain for the three-body vertex function the normalization condition of exactly the same form

$$1 = \langle\Gamma| \frac{\partial G_{BS}^0}{\partial P^2} |\Gamma\rangle - \langle\Gamma| G_{BS}^0 \frac{\partial \mathcal{V}}{\partial P^2} G_{BS}^0 |\Gamma\rangle = -\langle\Gamma| G_{BS}^0 \frac{\partial}{\partial P^2} (\mathcal{V} G_{BS}^0) |\Gamma\rangle. \quad (3.15)$$

The two forms of the normalization condition (3.15) are identical because of the bound state Eq. (3.14).

2. Identical particles

The equations and normalization condition for identical particles can be obtained immediately by symmetrizing the results given above. However, to lay the foundation for the

discussion of the spectator equations it is convenient to describe this symmetrization in detail. Most of the operator algebra below is exactly the same as in non-relativistic case.

The normalized three-body antisymmetrization operator is

$$\begin{aligned}
\mathcal{A}_3 &\equiv \frac{1}{3!} (1 + \zeta\mathcal{P}_{12} + \zeta\mathcal{P}_{13} + \zeta\mathcal{P}_{23} + \mathcal{P}_4 + \mathcal{P}_5) \\
&= \frac{1}{3!} (1 + \zeta\mathcal{P}_{ij})(1 + \zeta\mathcal{P}_{ik} + \zeta\mathcal{P}_{jk}) \\
&= \frac{1}{3!} (1 + \zeta\mathcal{P}_{ik} + \zeta\mathcal{P}_{jk})(1 + \zeta\mathcal{P}_{ij}) ,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
\mathcal{P}_4 &= \mathcal{P}_{13}\mathcal{P}_{12} = \mathcal{P}_{23}\mathcal{P}_{13} = \mathcal{P}_{12}\mathcal{P}_{23} \\
\mathcal{P}_5 &= \mathcal{P}_{23}\mathcal{P}_{12} = \mathcal{P}_{12}\mathcal{P}_{13} = \mathcal{P}_{13}\mathcal{P}_{23} .
\end{aligned} \tag{3.17}$$

Note that $\mathcal{A}_3 = \mathcal{A}_3\mathcal{A}_2 = \mathcal{A}_2\mathcal{A}_3$ where we may choose to have the two body symmetrization operator \mathcal{A}_2 act on *any* of the three two-body subspaces. We will exploit this property in the following discussion.

Using the relations $\mathcal{P}_{jk}V^i\mathcal{P}_{jk} = V^i$ and $\mathcal{P}_{ji}V^i\mathcal{P}_{ij} = V^j$ (which also hold for G_i) we can readily show that the symmetrization operator \mathcal{A}_3 commutes with the total potential (3.3). The symmetrized version of Eq. (3.1) is then obtained by multiplying the equation by \mathcal{A}_3 , which gives

$$\begin{aligned}
T &= \mathcal{A}_3\mathcal{T} = \mathcal{A}_3\mathcal{V} - \mathcal{A}_3\mathcal{V}G_{BS}^0\mathcal{T} \\
&= \mathcal{A}_3\mathcal{V} - \mathcal{A}_3\mathcal{V}G_{BS}^0\mathcal{A}_3\mathcal{T} \\
&= \bar{\mathcal{V}} - \bar{\mathcal{V}}G_{BS}^0T ,
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
\bar{\mathcal{V}} &= \mathcal{A}_3\mathcal{V} = \mathcal{A}_3\sum_i \bar{\mathcal{V}}^i \\
\bar{\mathcal{V}}^i &= i\bar{\mathcal{V}}^i G_i^{-1} .
\end{aligned} \tag{3.19}$$

A similar argument works for the transposed equation (3.2), and hence the symmetrized amplitude T satisfies the equations satisfied by \mathcal{T} with \mathcal{V} replaced by $\bar{\mathcal{V}}$. Note that we have written the kernel (3.19) in terms of the symmetrized two-body kernels (2.17), and that $\bar{\mathcal{V}}$ can also be written in terms of a single $\bar{\mathcal{V}}^i$

$$\bar{\mathcal{V}} = \mathcal{A}_3\sum_{\ell} \bar{\mathcal{V}}^{\ell} = \mathcal{A}_3\sum_{\ell} \mathcal{P}_{\ell i}\bar{\mathcal{V}}^i\mathcal{P}_{i\ell} = \mathcal{A}_3\bar{\mathcal{V}}^i\sum_{\ell} \zeta\mathcal{P}_{i\ell} . \tag{3.20}$$

As before, ℓ can be any index, and consistency requires the definition $\mathcal{P}_{ii} = \zeta$, so that $\zeta\mathcal{P}_{ii} = 1$. The last step in (3.20) follows from $\mathcal{P}_{ji}\bar{\mathcal{V}}^i\mathcal{P}_{ij} = \bar{\mathcal{V}}^j$ and $\mathcal{A}_3\mathcal{P}_{i\ell} = \zeta\mathcal{A}_3$. The symmetric version of Eq. (3.6)

$$\bar{\mathcal{V}}^i G_{BS}^0 = \bar{\mathcal{V}}^i G_{BS}^i . \tag{3.21}$$

will be useful in the following discussion.

The symmetrized bound state vertex function $|\Gamma\rangle_s = \mathcal{A}_3|\Gamma\rangle = \mathcal{A}_3\sum_i|\Gamma^i\rangle$ satisfies the equations

$$\begin{aligned} |\Gamma\rangle_s &= -\bar{\mathcal{V}}G_{BS}^0|\Gamma\rangle_s = -3\mathcal{A}_3\bar{\mathcal{V}}^i G_{BS}^i|\Gamma\rangle_s \\ {}_s\langle\Gamma| &= -{}_s\langle\Gamma|G_{BS}^0\bar{\mathcal{V}} = -3{}_s\langle\Gamma|G_{BS}^i\bar{\mathcal{V}}^i\mathcal{A}_3, \end{aligned} \quad (3.22)$$

where we used

$$\begin{aligned} \mathcal{P}_{ij}|\Gamma\rangle_s &= \zeta|\Gamma\rangle_s \\ \sum_{\ell} \zeta\mathcal{P}_{i\ell}|\Gamma\rangle_s &= 3|\Gamma\rangle_s. \end{aligned} \quad (3.23)$$

The normalization condition for this vertex function follows readily from the previous derivation

$$1 = {}_s\langle\Gamma|\frac{\partial G_{BS}^0}{\partial P^2}|\Gamma\rangle_s - {}_s\langle\Gamma|G_{BS}^0\frac{\partial\bar{\mathcal{V}}}{\partial P^2}G_{BS}^0|\Gamma\rangle_s = -{}_s\langle\Gamma|G_{BS}^0\frac{\partial}{\partial P^2}(\bar{\mathcal{V}}G_{BS}^0)|\Gamma\rangle_s. \quad (3.24)$$

Note that this equation has a structure identical to all of the previous results.

To make a better comparison with the spectator equations, it is convenient to obtain the BS equations and normalization condition for the symmetrized subamplitudes. First introduce the operator

$$\mathcal{A}_{ii'} = \frac{1}{3!}(1 + \zeta\mathcal{P}_{jk})\zeta\mathcal{P}_{ii'} = \frac{1}{3!}\zeta\mathcal{P}_{ii'}(1 + \zeta\mathcal{P}_{j'k'}). \quad (3.25)$$

The two forms of $\mathcal{A}_{ii'}$ are identical regardless of the specific values of i and i' . If $i = i'$, then $\mathcal{P}_{ii'} = \zeta$ and $\mathcal{P}_{jk} = \mathcal{P}_{j'k'}$ because both sets of indices $\{ijk\}$ and $\{i'j'k'\}$ must assume the values 1, 2, and 3. If $i \neq i'$, then i' must equal either j or k . For definiteness assume $i' = j$. Then we may use the identity

$$\mathcal{P}_{jk}\mathcal{P}_{ij} = \mathcal{P}_{ij}\mathcal{P}_{ik} \quad (3.26)$$

to prove the equivalence. An identical argument works if $i' = k$. These identities will be used frequently in the discussion which follows. Note that

$$\sum_i \mathcal{A}_{ii'} = \mathcal{A}_3 = \sum_{i'} \mathcal{A}_{ii'}. \quad (3.27)$$

Now we define the symmetrized subamplitudes $T^{ii'}$ and $|\Gamma^i\rangle_s$

$$T^{ii'} = \sum_{\ell\ell'} \mathcal{A}_{i\ell} \mathcal{T}^{\ell\ell'} \mathcal{A}_{\ell'i'} \quad (3.28)$$

$$|\Gamma^i\rangle_s = \sum_{\ell} \mathcal{A}_{i\ell} |\Gamma^{\ell}\rangle. \quad (3.29)$$

Using (3.27) gives the relations

$$\sum_i T^{ii'} = \mathcal{A}_3 \sum_{\ell\ell'} \mathcal{T}^{\ell\ell'} \mathcal{A}_{\ell'i'} \quad (3.30)$$

$$\sum_{ii'} T^{ii'} = \mathcal{A}_3 \sum_{\ell\ell'} \mathcal{T}^{\ell\ell'} \mathcal{A}_3 = \mathcal{A}_3 \mathcal{T} \mathcal{A}_3 = T \quad (3.31)$$

$$\sum_i |\Gamma^i\rangle_s = \mathcal{A}_3 \sum_{\ell} |\Gamma^\ell\rangle = \mathcal{A}_3 |\Gamma\rangle = |\Gamma\rangle_s. \quad (3.32)$$

From the definitions it follows immediately that

$$\mathcal{P}_{jk} T^{ii'} = \zeta T^{ii'} = T^{ii'} \mathcal{P}_{j'k'} \quad (3.33)$$

$$\mathcal{P}_{jk} |\Gamma^i\rangle_s = \zeta |\Gamma^i\rangle_s. \quad (3.34)$$

It also follows from (3.25) that

$$\begin{aligned} \mathcal{P}_{ij} T^{ii'} &= \zeta T^{ji'} \\ T^{ii'} \mathcal{P}_{i'j'} &= \zeta T^{ij'} \\ \mathcal{P}_{ij} |\Gamma^i\rangle_s &= \zeta |\Gamma^j\rangle_s. \end{aligned} \quad (3.35)$$

To prove these relations define, for some operators $\mathcal{O}^1, \mathcal{O}^2, \mathcal{O}^3$, the operator $\tilde{\mathcal{O}}^i$

$$\tilde{\mathcal{O}}^i = \sum_{\ell} \mathcal{A}_{i\ell} \mathcal{O}^{\ell}. \quad (3.36)$$

Then

$$\begin{aligned} \mathcal{P}_{ij} \tilde{\mathcal{O}}^i &= \frac{1}{6} \mathcal{P}_{ij} (1 + \zeta \mathcal{P}_{jk}) \sum_{\ell} \zeta \mathcal{P}_{i\ell} \mathcal{O}^{\ell} \\ &= \frac{1}{6} (1 + \zeta \mathcal{P}_{ik}) \mathcal{P}_{ij} (\mathcal{O}^i + \zeta \mathcal{P}_{ij} \mathcal{O}^j + \zeta \mathcal{P}_{ik} \mathcal{O}^k) \\ &= \frac{1}{6} (1 + \zeta \mathcal{P}_{ik}) (\mathcal{P}_{ij} \mathcal{O}^i + \zeta \mathcal{O}^j + \zeta \mathcal{P}_{ik} \mathcal{P}_{jk} \mathcal{O}^k) \\ &= \zeta \tilde{\mathcal{O}}^j. \end{aligned} \quad (3.37)$$

In the last step we used $(1 + \zeta \mathcal{P}_{ik}) \mathcal{P}_{ik} = \zeta (1 + \zeta \mathcal{P}_{ik})$. The first of the relations (3.35) follows immediately from (3.37), and the second by taking the hermitian conjugate.

Next, note that

$$\mathcal{A}_{i\ell} \mathcal{V}^{\ell} = \mathcal{A}_{i\ell} \bar{\mathcal{V}}^{\ell} = \bar{\mathcal{V}}^i \mathcal{A}_{i\ell}, \quad (3.38)$$

and

$$\begin{aligned} \sum_{\ell\ell'} \mathcal{A}_{i\ell} \delta_{\ell\ell'} \mathcal{V}^{\ell} \mathcal{A}_{\ell'i'} &= \frac{1}{9} \sum_{\ell} \mathcal{P}_{i\ell} \bar{\mathcal{V}}^{\ell} \mathcal{P}_{\ell i'} \\ &= \frac{1}{9} \bar{\mathcal{V}}^i \sum_{\ell} \mathcal{P}_{i\ell} \mathcal{P}_{\ell i'} = \zeta \frac{1}{3} \bar{\mathcal{V}}^i \mathcal{P}_{ii'}, \end{aligned} \quad (3.39)$$

where in the last step we have used

$$\frac{1}{2}(1 + \zeta P_{jk}) \sum_{\ell} \mathcal{P}_{i\ell} \mathcal{P}_{\ell i'} = \frac{1}{2}(1 + \zeta P_{jk}) 3\zeta P_{i i'}. \quad (3.40)$$

Starting from Eqs. (3.4) or (3.5), using the definition (3.28) and the properties (3.39), (3.38), (3.30), and (3.27) gives the following equations for the symmetrized subamplitudes

$$T^{ii'} = i \zeta \frac{1}{3} \bar{V}^i G_i^{-1} \mathcal{P}_{i i'} - \bar{V}^i G_{BS}^i \sum_{\ell} T^{\ell i'} \quad (3.41)$$

$$= i \zeta \frac{1}{3} \mathcal{P}_{i i'} \bar{V}^{i'} G_i^{-1} - \sum_{\ell} T^{i \ell} G_{BS}^{i'} \bar{V}^{i'}. \quad (3.42)$$

Summing Eq. (3.41) over i and i' recovers Eq. (3.18), demonstrating consistency. Note that Eq. (3.41) is equivalent to

$$T^{ii} = i \frac{1}{3} \bar{V}^i G_i^{-1} - \bar{V}^i G_{BS}^i \sum_{\ell} T^{\ell i} \quad (3.43)$$

$$T^{i i'} = \zeta T^{i i'} \mathcal{P}_{i i'}, \quad (3.44)$$

where (3.44) merely recovers the symmetry (3.35). Using this symmetry again, we may reduce (3.43) to an equation for the single amplitude T^{ii}

$$\begin{aligned} T^{ii} &= i \frac{1}{3} \bar{V}^i G_i^{-1} - \bar{V}^i G_{BS}^i (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) T^{ii} \\ &= i \frac{1}{3} \bar{V}^i G_i^{-1} - \bar{V}^i G_{BS}^i (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{jk} \mathcal{P}_{ij} \mathcal{P}_{jk}) T^{ii} \\ &= i \frac{1}{3} \bar{V}^i G_i^{-1} - \bar{V}^i G_{BS}^i (1 + 2\zeta \mathcal{P}_{ij}) T^{ii}. \end{aligned} \quad (3.45)$$

Hence it is sufficient to solve one equation for $i = i'$, and obtain all other amplitudes from (3.44). Using Eq. (2.18) for the symmetrized two-body scattering amplitude, the equation for T^{ii} may be further reduced to

$$T^{ii} = i \frac{1}{3} M^i G_i^{-1} - 2\zeta M^i G_{BS}^i \mathcal{P}_{ij} T^{ii}. \quad (3.46)$$

Similarly, the equations for the symmetrized subvertex functions are

$$|\Gamma^i\rangle_s = -\bar{V}^i G_{BS}^i (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma^i\rangle_s = -\bar{V}^i G_{BS}^i |\Gamma\rangle_s \quad (3.47)$$

$$= -2\zeta M^i G_{BS}^i \mathcal{P}_{ij} |\Gamma^i\rangle_s. \quad (3.48)$$

Using the definitions of the symmetric subvertex functions, the normalization condition (3.24) can be further simplified. First note that

$$\frac{\partial}{\partial P^2} (\bar{V} G_{BS}^0) = \mathcal{A}_3 \frac{\partial}{\partial P^2} (\bar{V}^i G_{BS}^i) \sum_{\ell} \zeta \mathcal{P}_{i\ell} = \mathcal{A}_3 \left\{ \frac{\partial \bar{V}^i}{\partial P^2} G_{BS}^i + \bar{V}^i \frac{\partial G_{BS}^i}{\partial P^2} \right\} \sum_{\ell} \zeta \mathcal{P}_{i\ell}. \quad (3.49)$$

To obtain this we used Eqs. (3.20,3.21) and the fact that the exchange operators \mathcal{P}_{ij} are independent of P^2 , so that $\partial \mathcal{P}_{ij} / \partial P^2 = 0$ for any exchange operator. Substituting this

relation into (3.24) and using the bound state equation (3.47) and $\mathcal{A}_3|\Gamma\rangle_s = |\Gamma\rangle_s$ gives immediately

$$1 = -3i \langle \Gamma^i | G_i \frac{\partial G_{BS}^i}{\partial P^2} |\Gamma\rangle_s + 3i \langle \Gamma | G_{BS}^i G_i \frac{\partial \bar{V}^i}{\partial P^2} G_{BS}^i |\Gamma\rangle_s. \quad (3.50)$$

This condition can be expressed in terms of a single subvertex function, which we will choose to be $|\Gamma^i\rangle_s$. Denoting the operator $G_{BS}^i G_i (\partial \bar{V}^i / \partial P^2) G_{BS}^i$ by \mathcal{O}_2^i , and remembering that this operator contains a factor of $1 + \zeta \mathcal{P}_{jk}$, the second term reduces to

$$\begin{aligned} \langle \Gamma | \mathcal{O}_2^i |\Gamma\rangle_s &= \langle \Gamma^i | (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) \mathcal{O}_2^i (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) |\Gamma^i\rangle_s \\ &= \langle \Gamma^i | (1 + 2\zeta \mathcal{P}_{ij}) \mathcal{O}_2^i (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma^i\rangle_s \end{aligned} \quad (3.51)$$

where we used $\mathcal{P}_{jk} \mathcal{P}_{ij} \mathcal{P}_{jk} = \mathcal{P}_{ik} \rightarrow \mathcal{P}_{ij}$ as the operators \mathcal{P}_{jk} are eliminated using (3.34) and $\mathcal{P}_{jk}^2 = 1$. Similarly, since the operator \mathcal{P}_{jk} commutes with $\mathcal{O}_1^i = G_i (\partial G_{BS}^i / \partial P^2)$, the first term becomes

$$\begin{aligned} \langle \Gamma^i | \mathcal{O}_1^i |\Gamma\rangle_s &= \langle \Gamma^i | \mathcal{O}_1^i (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) |\Gamma^i\rangle_s \\ &= \langle \Gamma^i | \mathcal{O}_1^i (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma^i\rangle_s. \end{aligned} \quad (3.52)$$

Combining these terms gives a normalization condition expressed in terms of $|\Gamma^i\rangle_s$

$$\begin{aligned} 1 &= -3i \langle \Gamma^i | G_i \frac{\partial G_{BS}^i}{\partial P^2} (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma^i\rangle_s \\ &\quad + 3i \langle \Gamma^i | (1 + 2\zeta \mathcal{P}_{ij}) G_{BS}^i G_i \frac{\partial \bar{V}^i}{\partial P^2} G_{BS}^i (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma^i\rangle_s. \end{aligned} \quad (3.53)$$

We are now ready to turn our discussion to the three-body spectator equation.

B. Normalization of the Three-body Gross Vertex Functions

The three-body spectator or Gross equation for identical particles is obtained from the symmetrized three-body Bethe-Salpeter equation (3.46) by placing the spectator particle on its positive energy mass shell [4,6]. In order to obtain a closed set of coupled equations a second particle must be placed on its mass shell in the initial and final states, as illustrated in Fig. 9. Therefore, we label those particles that are on-shell by replacing subamplitudes $T^{ii'}$ by $T^{ii'} \rightarrow T_{jj'}^{ii'} = \mathcal{Q}_i \mathcal{Q}_j T^{ii'} \mathcal{Q}_{i'} \mathcal{Q}_{j'}$. Noting that $\mathcal{P}_{jj'} \mathcal{Q}_j \mathcal{P}_{jj'} = \mathcal{Q}_{j'}$, and using Eqs. (3.33) and (3.35) we have

$$\begin{aligned} \mathcal{P}_{ij} T_{jj'}^{ii'} &= \mathcal{P}_{ij} \mathcal{Q}_i \mathcal{Q}_j T^{ii'} \mathcal{Q}_{i'} \mathcal{Q}_{j'} = \mathcal{Q}_j \mathcal{Q}_i \zeta T^{ji'} \mathcal{Q}_{i'} \mathcal{Q}_{j'} = \zeta T_{ij'}^{ji'} \\ \mathcal{P}_{ik} T_{jj'}^{ii'} &= \mathcal{Q}_k \mathcal{Q}_j \mathcal{P}_{ik} T^{ii'} \mathcal{Q}_{i'} \mathcal{Q}_{j'} = \zeta T_{jj'}^{ki'} \\ \mathcal{P}_{jk} T_{jj'}^{ii'} &= \mathcal{Q}_i \mathcal{Q}_k \mathcal{P}_{jk} T^{ii'} \mathcal{Q}_{i'} \mathcal{Q}_{j'} = \zeta T_{kj'}^{ii'}. \end{aligned} \quad (3.54)$$

Similar relations exist for the initial state. Hence any amplitude can be obtained by the action of the permutation operators on the canonical amplitude T_{jj}^{ii} , and it is necessary

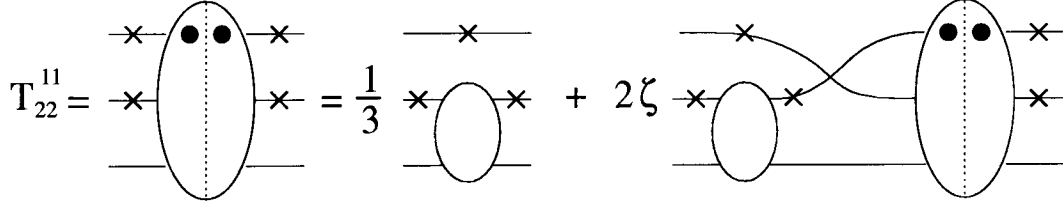


FIG. 9. Diagrammatic representation of the Gross equation for the amplitude T_{22}^{11} . On-shell particles are labeled with an \times . The final state particles emerge from the left side of each diagram, and putting the final state spectator on-shell consistently automatically forces a second particle in the final state to be on-shell. In this way two particles are always on-shell.

only to find and solve the equation for this one amplitude. Note that we can choose i and j arbitrarily as long as $i \neq j$. The subamplitudes $T_{jj'}^{ii'}$ cannot be added unless the same pairs of particles are on shell in both the initial and final states. Therefore, there is no total scattering amplitude such as (3.7) for the 3-body spectator equation. The underlying physical justification for the use of the spectator equations is described in detail in Refs. [4,6]. We will show how to use the subamplitudes $T_{jj'}^{ii'}$ in constructing the matrix elements in a forthcoming paper [13].

The equations for $T_{jj'}^{ii'}$ were first derived in [4] in terms of the two-body scattering matrices $M_{jj'}^i$, by analysis of ladder and crossed ladder exchanges. Formally they follow from the three-body equation in the Bethe-Salpeter framework by appropriate restriction of the propagation of the on-shell particles. Thus, equations for T_{jj}^{ii} can be obtained from Eq. (3.46) by replacing the two-body propagator G_{BS}^i by G_j^i , where $G_j^i = G_k \mathcal{Q}_j$. The inverse propagator in the inhomogeneous term, whose role was to cancel one of the spectator propagators when the equations are iterated, must be replaced by unity, $iG_i^{-1} \rightarrow 1$, since the spectator is always on shell. This gives the following equations for T_{jj}^{ii}

$$T_{jj}^{ii} = \frac{1}{3} M_{jj}^i - 2\zeta M_{jj}^i G_j^i \mathcal{P}_{ij} T_{jj}^{ii} \quad (3.55)$$

$$= \frac{1}{3} M_{jj}^i - 2\zeta T_{jj}^{ii} \mathcal{P}_{ij} G_j^i M_{jj}^i \quad (3.56)$$

where the two body scattering operator $M_{jj}^i = \mathcal{Q}_j M^i \mathcal{Q}_j$ satisfies

$$M_{jj}^i = \bar{V}_{jj}^i - \bar{V}_{jj}^i G_j^i M_{jj}^i \quad (3.57)$$

$$M_{jj}^i = \bar{V}_{jj}^i - M_{jj}^i G_j^i \bar{V}_{jj}^i, \quad (3.58)$$

Eq. (3.55) is illustrated diagrammatically in Fig. 9. In terms of the kernels \bar{V}_{jj}^i this equation becomes

$$T_{jj}^{ii} = \frac{1}{3} \bar{V}_{jj}^i - \bar{V}_{jj}^i G_j^i (1 + 2\zeta \mathcal{P}_{ij}) T_{jj}^{ii} \quad (3.59)$$

$$= \frac{1}{3} \bar{V}_{jj}^i - T_{jj}^{ii} (1 + 2\zeta \mathcal{P}_{ij}) G_j^i \bar{V}_{jj}^i. \quad (3.60)$$

Subvertex functions acquire an additional index to denote the second on-shell particle, $|\Gamma^i\rangle \rightarrow |\Gamma_j^i\rangle = \mathcal{Q}_i \mathcal{Q}_j |\Gamma^i\rangle$, and the symmetrized subvertex functions satisfy the following equations

$$|\Gamma_j^i\rangle = -2\zeta M_{jj}^i G_j^i \mathcal{P}_{ij} |\Gamma_j^i\rangle \quad (3.61)$$

$$= -\bar{V}_{jj}^i G_j^i (1 + 2\zeta \mathcal{P}_{ij}) |\Gamma_j^i\rangle. \quad (3.62)$$

Derivation of the normalization condition proceeds as for the Bethe-Salpeter case, but for clarity we will present the full derivation. Obtaining \bar{V}_{jj}^i from Eq. (3.60), and substituting this into Eq. (3.59) gives the following nonlinear equation for T_{jj}^{ii}

$$T_{jj}^{ii} = \frac{1}{3} \bar{V}_{jj}^i - 3T_{jj}^{ii} G_j^i \mathcal{S}_{ij} T_{jj}^{ii} - 3T_{jj}^{ii} \mathcal{S}_{ij} G_j^i \bar{V}_{jj}^i G_j^i \mathcal{S}_{ij} T_{jj}^{ii}, \quad (3.63)$$

where $\mathcal{S}_{ij} = (1 + 2\zeta \mathcal{P}_{ij})$. The existence of the bound state implies that

$$T_{jj}^{ii} = -\frac{|\Gamma_j^i\rangle \langle \Gamma_j^i|}{M^2 - P^2} + R_{jj}^{ii}, \quad (3.64)$$

where, as before, R_{jj}^{ii} is regular at the pole at $P^2 = M^2$. Requiring the coefficient of the double pole on the rhs of this equation to be zero gives the bound state equation (3.62). Equating the coefficients of the single pole terms gives an equation similar to (2.8):

$$\begin{aligned} |\Gamma_j^i\rangle \langle \Gamma_j^i| = & \mathfrak{Z} \lim_{P^2 \rightarrow M^2} \left\{ |\Gamma_j^i\rangle \left[\frac{\langle \Gamma_j^i| (1 + \mathcal{S}_{ij} G_j^i \bar{V}_{jj}^i) G_j^i \mathcal{S}_{ij} |\Gamma_j^i\rangle}{M^2 - P^2} \right] \langle \Gamma_j^i| \right. \\ & - \mathcal{R}_{jj}^{ii} \mathcal{S}_{ij} G_j^i (1 + \bar{V}_{jj}^i G_j^i \mathcal{S}_{ij}) |\Gamma_j^i\rangle \langle \Gamma_j^i| \\ & \left. - |\Gamma_j^i\rangle \langle \Gamma_j^i| (1 + \mathcal{S}_{ij} G_j^i \bar{V}_{jj}^i) G_j^i \mathcal{S}_{ij} \mathcal{R}_{jj}^{ii} \right\}, \quad (3.65) \end{aligned}$$

where we used the fact that \mathcal{S}_{ij} commutes with $\mathcal{Q}_i G_j^i$. The bound state equation ensures that the terms involving \mathcal{R}_{jj}^{ii} and any terms coming from the derivatives of the subvertex functions $|\Gamma_j^i\rangle$ with respect to P^2 are zero. Since $\partial \mathcal{S}_{ij} / \partial P^2$ is also zero, the condition (3.65) reduces to

$$1 = -3 \left\{ \langle \Gamma_j^i | \mathcal{S}_{ij} \frac{\partial G_j^i}{\partial P^2} | \Gamma_j^i \rangle + \langle \Gamma_j^i | \mathcal{S}_{ij} \frac{\partial}{\partial P^2} (G_j^i \bar{V}_{jj}^i G_j^i) \mathcal{S}_{ij} | \Gamma_j^i \rangle \right\}. \quad (3.66)$$

Distributing the derivative over the second term and using the bound state equation gives our final result for the normalization condition for the subamplitude $|\Gamma_j^i\rangle$

$$1 = 3 \left\{ \langle \Gamma_j^i | (1 + 2\zeta \mathcal{P}_{ij}) (G_j^i)' | \Gamma_j^i \rangle - \langle \Gamma_j^i | (1 + 2\zeta \mathcal{P}_{ij}) G_j^i (\bar{V}_{jj}^i)' G_j^i (1 + 2\zeta \mathcal{P}_{ij}) | \Gamma_j^i \rangle \right\}. \quad (3.67)$$

As in the two body case, this equation could have been obtained directly from Eq. (3.53).

It is interesting to compare our principal result (3.67) with the nonrelativistic normalization condition. If $|\Psi\rangle = \sum_i |\psi^i\rangle$ is the total three body wave function, and we choose to express the normalization in terms of the $i = 1$ component, the normalization condition is

$$\begin{aligned}
1 &= \langle \Psi | \Psi \rangle = \langle \Psi | (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) | \psi^i \rangle \\
&= 3 \langle \Psi | \psi^i \rangle = 3 \langle \psi^i | (1 + \zeta \mathcal{P}_{ij} + \zeta \mathcal{P}_{ik}) | \psi^i \rangle \\
&= 3 \langle \psi^i | (1 + 2\zeta \mathcal{P}_{ij}) | \psi^i \rangle = 3 \langle \psi^1 | (1 + 2\zeta \mathcal{P}_{12}) | \psi^1 \rangle.
\end{aligned} \tag{3.68}$$

This is very similar to the normalization condition for the Gross vertex function, Eq. (3.67). Adopting the convention that $i = 1$ and $j = 2$, using the fermion propagator given in Eq. (2.30), with $k_3 = P - k_1 - k_2$, gives (in the rest system)

$$\left[G_2^1(k_3) \right]' = G_3'(k_3) = G_3(k_3) \frac{P}{2P^2} G_3(k_3) = \frac{1}{2M_B} G_3(k_3) \gamma^0 G_3(k_3). \tag{3.69}$$

Neglecting the terms involving the derivative of the kernel in Eq. (3.67), decomposing the propagator into positive and negative energy parts,

$$G_3(k_3) = \left(\frac{1}{2E_{k_3}} \right) \sum_{\lambda} \left[\frac{u(\mathbf{k}_3, \lambda) \bar{u}(\mathbf{k}_3, \lambda)}{E_{k_1} + E_{k_2} + E_{k_3} - M_B} - \frac{v(-\mathbf{k}_3, \lambda) \bar{v}(-\mathbf{k}_3, \lambda)}{E_{k_3} + M_B - E_{k_1} - E_{k_2}} \right], \tag{3.70}$$

keeping only the positive energy part and using $\bar{u}(\mathbf{k}_3, \lambda') \gamma^0 u(\mathbf{k}_3, \lambda) = 2E_{k_3} \delta_{\lambda'\lambda}$, allows us to reduce the normalization condition (3.67) for the bound state vertex function for three identical fermions to

$$1 \simeq 3 \sum_{\lambda} \langle \psi_{\lambda}^1 | (1 + 2\zeta \mathcal{P}_{12}) | \psi_{\lambda}^1 \rangle, \tag{3.71}$$

where the wave function is related to the vertex function by

$$|\psi_{\lambda}^1 \rangle \simeq \frac{1}{\sqrt{2M_B}} \frac{u(\mathbf{k}_3, \lambda) |\Gamma_2^1 \rangle}{E_{k_1} + E_{k_2} + E_{k_3} - M_B}. \tag{3.72}$$

This correspondence will be developed in greater detail elsewhere. Here we only wish to emphasize that the covariant normalization condition (3.67) reduces to the expected non-relativistic limit.

IV. CONCLUSION

The principal results of this paper are the derivation of the normalization conditions (3.24) and (3.53) for the three-body BS vertex functions, and (3.67) for the three-body Gross vertex functions. These results are new.

In the course of deriving these results we also showed that the normalization condition for the two-body Gross equation is identical to the requirement that the charge of a bound state be equal to sum of the charges of its constituents. Since we derived the normalization condition without any reference to the charge, this is equivalent to a proof that charge is automatically conserved by the spectator theory. This result has been used in previous work [9], but the proof is new. A similar proof should be possible for the three body system, and will be presented in a subsequent paper [13].

ACKNOWLEDGMENTS

We are happy to acknowledge the support of the DOE through Jefferson Laboratory and grant No. DE-FG05-88ER40435.

APPENDIX: FEYNMAN RULES

In this paper, we use the Feynman rules as given in [15].

The variation from the usual field theoretical practice is with regard to the normalization of the symmetrized two- and three-body scattering matrices M and T . The practice in most field theory texts is to use an unnormalized symmetrization operator and to include all information about the symmetry of the scattering process in an overall phase space factor. Here we choose to use normalized symmetrization operators \mathcal{A}_2 for the two-body case and \mathcal{A}_3 for the three-body case, as defined by (3.16). This results in integral equations for the scattering amplitudes which are consistently of the form of the nonrelativistic Lippmann-Schwinger equations and preserves the nonrelativistic normalization of the wave functions. Any matrix element that involves an occurrence of the symmetrized scattering matrices as used here can be used in the usual expression for cross sections by making the substitutions $M \rightarrow 2M$ and $T \rightarrow 6T$.

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FIGURE CAPTIONS

Figure 1: Diagrammatic representation of the two-body BS equation for the scattering matrix.

Figure 2: Diagrammatic representation of the symmetrized kernel.

Figure 3: Diagrammatic representation of the two-body Gross equation for the scattering matrix. The \times on the line for particle 1 indicates that it is on shell.

Figure 4: Diagrammatic representation of the symmetrized kernel for Gross equation. As in Fig. 3, the \times indicates that the particle is on shell.

Figure 5: Diagrammatic representation of the normalization condition for the two-body Gross vertex function. The dashed line represents the derivative $\partial/\partial P^2$, and the \times means that the particle is on shell.

Figure 6: Diagrams which contribute to the electromagnetic charge of the two body bound state.

Figure 7: Location in the p_{10} complex plane of the 6 poles from the three nucleon propagators in Eq. (2.34). The last two terms in Eq. (2.33) emerge if the integral of p_{10} over the contour C enclosing the poles 3a and 1a is evaluated using the residue theorem.

Figure 8: Diagrammatic representation of the Faddeev equations for the amplitudes \mathcal{T}^{1i} . Note that the spectator is identified by the solid dot.

Figure 9: Diagrammatic representation of the Gross equation for the amplitude T_{22}^{11} . On-shell particles are labeled with an \times . The final state particles emerge from the left side of each diagram, and putting the final state spectator on-shell consistently automatically forces a second particle in the final state to be on-shell. In this way two particles are always on-shell.